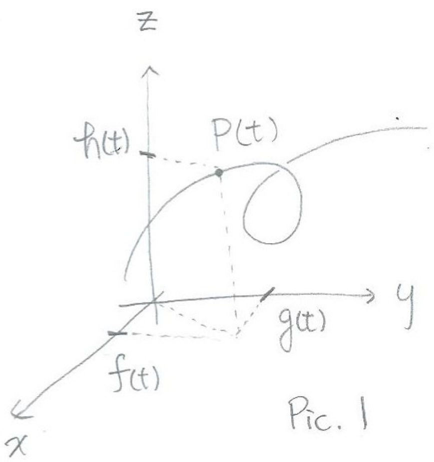
§ Curves and their tangents

Let $t \in \mathbb{R}$ be time variable. For each t, we can define a position function in $\mathbb{R}^3$:

$$P(t) = (f(t), g(t), h(t)) \quad \text{See. Pic. 1}$$



Pic. 1

t : parameter

If $f(t), g(t), h(t)$ are continuous, then $C = \{P(t) \mid t \in \mathbb{R}\}$ is a continuous curve.

Again, for each curve $C \subset \mathbb{R}^3$, the choice of parametrization is <u>not</u> unique.

<u>Define</u> $\vec{v}(t) = (f'(t), g'(t), h'(t))$ to be the velocity (tangent vector) of P.

$|\vec{v}|(t) = |\vec{v}(t)|$ to be the speed of P.

By fundamental theorem of calculus:

$$P(0) + \int_0^t \vec{v}(t)\, dt = P(t) \quad : \text{position at time } t.$$

where $\int_0^t \vec{v}(t)\, dt = \int_0^t (f'(t), g'(t), h'(t))\, dt$

$$=\left(\int_0^t f'(t)\,dt\ ,\ \int_0^t g'(t)\,dt\ ,\ \int_0^t h'(t)\,dt\right).$$

Generally, we call the type of function $\begin{cases} U(t) \\ P(t) \end{cases}$ a vector value function. It will satisfy the following properties:

Properties:

- $(r(t)\ \vec{v}(t))' = r'(t)\cdot\vec{v}(t) + r(t)(\vec{v}'(t))$

  where $r(t)$ is a scalar function.

- (Linearity). $(\vec{v}+c\vec{w})' = \vec{v}' + c\vec{w}'$

  for any two vector value function $\vec{v},\vec{w}$ and $c\in\mathbb{R}$.

- (Dot product) $(\vec{u}\cdot\vec{v})' = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$.
- (Cross product) $(\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

- (Chain rule):

$$(\vec{u}(r(t)))' = \frac{d}{dt}(\vec{u}(r(t))) = r'(t)\,\vec{u}'(r(t))$$

Arc length along a space curve:

Let $r(t) = (x(t), y(t), z(t))$

then the arc length from $r(a)$ to $r(b)$ is

$$L_{ab} = \int_a^b |r'(t)|\,dt$$

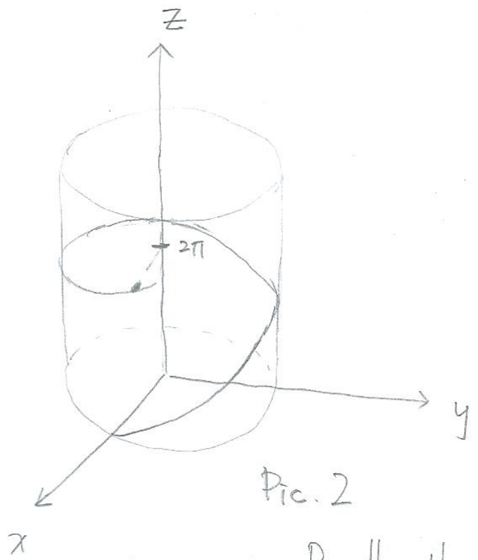
Example: (Helix)

$$r(t) = (\cos t, \sin t, t)$$

The arc length of $C = \{r(t)\}$ from $r(0)$ to $r(2\pi)$ is

$$L_{0,2\pi} = \int_0^{2\pi} |r'(t)|\,dt = \int_0^{2\pi} [(-\sin)^2(t) + \cos^2 t + 1]^{1/2}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\,\pi.$$



Pic. 2

Rmk • If we regard $r(t)$ to be the position function of a particle, then $|r'(t)|$ is the speed. The arc length intergral of $|r'(t)|$.

• Recall that the parametrizations of a curve is not unique. We can always choose one that $|r'(t)| \equiv 1$. In this case, the arc length

$$L_{ab} = \int_a^b 1\,dt = |b-a|.$$

Curvature: Consider a plane curve $r(t) = (x(t), y(t))$, with $|r'(t)| \equiv 1$. Then we define

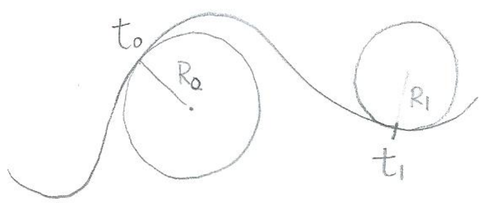
$$K(t) = |r''(t)|$$

Never forget the condition $|r'(t)| \equiv 1$.

<u>Geometric meaning:</u> $K = \frac{1}{R}$, where R is the radius of the approximation circle (circle of curvature, osculating circle)

See Pic 3

<u>Pf:</u>



Pic 3.

Let $C = \{C(t)\}$ be the osculating circle with

$$\begin{cases} |C'(t_0)| = 1 = |r'(t_0)| \\ |C''(t_0)| = K = |r''(t_0)| \\ C(t_0) = r(t_0) \quad \text{for some } t_0. \end{cases}$$

Since $C(t)$ is a circle, we can take

$$C(t) = \left(R\cos(\alpha t+\beta) + P_1 \,,\ R\sin(\alpha t+\beta) + P_2 \right)$$

Now, $|C'(t_0)| = 1 \Rightarrow \left[\alpha^2 R^2 \sin(\alpha t_0+\beta)^2 + \alpha^2 R^2 \cos(\alpha t_0+\beta)^2 \right]^{1/2}$

$$= |\alpha||R| = 1 \quad \therefore |\alpha| = \frac{1}{R}$$

$$\therefore |C''(t_0)| = \left[\alpha^4 R^2 \cos(\alpha t_0+\beta)^2 + \alpha^4 R^2 \sin(\alpha t_0+\beta)^2 \right]^{1/2}$$

$$= |\alpha^2 R| = \frac{1}{R}.$$

In particular, the curvature of a circle of radius R is a constant function $\kappa = \frac{1}{R}$.

<u>Normal vector:</u> For a plane curve $r(t) = (x(t), y(t))$, with $|r'(t)| = 1$, we have

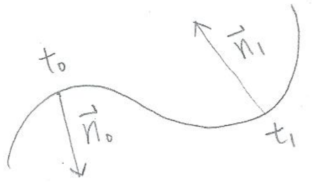
$$\frac{d}{dt}|r'(t)|^2 = \frac{d}{dt}(r'(t) \cdot r'(t))$$

$$= 2\, r''(t) \cdot r'(t) = 0$$

$\therefore$ $r''(t) \perp r'(t)$

So we call $\vec{n} = \dfrac{r''(t)}{|r''(t)|}$ the Principal unit normal of $r(t)$.

see. Pic. 4

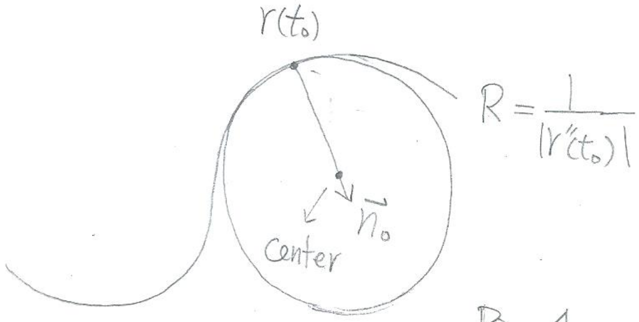


Now, the center of the osculating circle is on $\{ r(t_0) + s \cdot \vec{n}_0 ,\ \text{for } s > 0 \}$

So we can write down the eq:

$$C_{t_0} = \{ (\underbrace{\frac{1}{|r''(t_0)|}}_{= R} \cos(t) ,\ \frac{1}{|r''(t_0)|} \sin(t)) + \underbrace{r(t_0) + \frac{r''(t_0)}{|r''(t_0)|^2}}_{= \text{center}} \}$$

<u>See. Pic.4</u>



Pic. 4.

<u>Space curve: Normal & Binormal:</u>

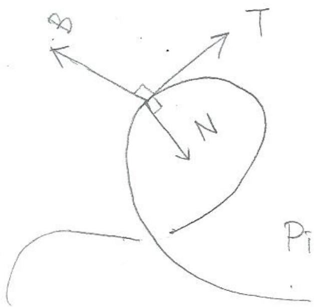
Let $r(t) = (x(t), y(t), z(t))$ be a space curve with $|r'(t)| \equiv 1$. We have

$T := r'(t)$: unit tangent vector

$N := \dfrac{r''(t)}{|r''(t)|}$: unit normal vector.

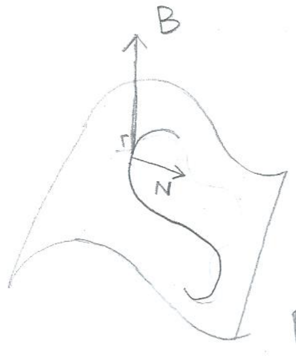
<u>Define</u>: $B := T \times N$ to be the Binormal vector see Pic. 5

$$\begin{cases} B \perp T \\ N \perp T \\ B \perp N \end{cases}$$



Pic. 5

Geometrically, if we have a plane curve on a sheet of paper. N will be on this paper and B will be perpendicular to this paper. see Pic. 6



Pic. 6.

Acceleration: Let $r(t) = (x(t), y(t), z(t))$ (Now we don't have $|r'(t)| = 1$)

Recall that $\vec{v} = \frac{dr}{dt}$: velocity

Change the parameter. $t \to s$ s.t $|r'(s)| = 1$, we have

$$T = \frac{dr}{ds} : \text{unit tangent vector.}$$

Acceleration :

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\overbrace{\left(\frac{dr}{ds} \cdot \frac{ds}{dt}\right)}^{\text{Chain rule}} = \frac{d}{dt}\left(T \cdot \frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2} \cdot T + \frac{dT}{dt} \frac{ds}{dt}$$

$$= \frac{d^2s}{dt^2} \cdot T + \frac{dT}{ds}\left(\frac{ds}{dt}\right)^2$$

$$= \frac{d^2s}{dt^2} \cdot T + \left(\frac{ds}{dt}\right)^2 \cdot KN$$

$$= a_T T + a_N \cdot N.$$

where $\begin{cases} a_T = \frac{d^2s}{dt^2} & \text{tangent scalar component of } \vec{a} \\ a_N = \left(\frac{ds}{dt}\right)^2 \cdot K & \text{normal scalar component of } \vec{a} \end{cases}$

No Binormal component.

How to find the parameter s s.t. $|r'(s)| = 1$?

Suppose we have $r(t) = (x(t), y(t), z(t))$.

let $t = t(s)$.

$$\left|\frac{d}{ds}(r(t(s)))\right| = \left|(x'(t)\cdot t'(s),\ y'(t)\cdot t'(s),\ z'(t)\cdot t'(s))\right|$$

$$= |t'(s)\, r'(t)| = 1$$

so $t'(s) = \dfrac{1}{|r'(t)|} = \dfrac{dt}{ds}$

$\Rightarrow \dfrac{ds}{dt} = |r'(t)|$, take $s = \int_0^t |r'(t)|$,

We have the parameter for s.

Namely, we should parametrize a curve by its arc length.